On The Symplectic Two-Form of Gravity in Terms of Dirac Eigenvalues

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Abstract

The Dirac eigenvalues form a subset of observables of the Euclidean gravity. The symplectic two-form in the covariant phase space could be expressed, in principle, in terms of the Dirac eigenvalues. We discuss the existence of the formal solution of the equations defining the components of the symplectic form in this framework.

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One of the major obstacles in quantizing the gravity is finding a complete set of observables of it. Recently, a certain progess has been made in defining and manipulating covariant observables in various theories of gravity [1, 2, 3, 4]. Previous works showed that the Dirac eigenvalues can be considered as observables of gravity, too, on manifolds endowed with an Euclidean structure [5, 6, 7]. The result was generalized to include the local $N = 1$ supersymmetry in [8, 9, 10, 11] and it was shown to be connected to spectral geometry in [12]. However, in order to completely understand the covariant phase space of the Euclidean gravity in terms of the Dirac eigenvalues, one has to know what is the form of the symplectic two-form in terms of the observables. The aim of this letter is to discuss the existence of a formal solution of this problem.

Let us begin by considering a four-dimensional compact manifold M without boundary endowed with an Euclidean metric field $g_{\mu\nu}$. One introduces a tetrad field which maps the metric at each point $x \in M$ to the local Euclidean metric in the tangent space: $g_{\mu\nu}(x) = E^I_\mu(x)E^J_\nu(x)\delta_{IJ}$. The covariant phase space of the theory is given by non-equivalent solutions of the Eistein equations on M modulo the "gauge transformations", i. e. transformations generated by local $SO(4)$ times diffeomorphisms. The functions of the phase space are observables of the theory. Consider now the Dirac equation

$$D \mid \psi_n\rangle = \lambda_n \mid \psi_n\rangle, \tag{1}$$

where $|\psi_n\rangle$ is a spinor field (in the Dirac's bra-ket notation) and n is a positive integer (for simplicity, we assume that the Dirac operator D has no zero eigenvalue.) The eigenvalues λ_n define a discrete family of real valued functions on the space of smooth tetrads $\mathcal{E}$ and a function from $\mathcal{E}$ into the space of infinite sequences R^∞

$$\lambda_n \quad : \quad \mathcal{E} \longrightarrow R \quad , \quad E \rightarrow \lambda_n[E], \tag{2}$$
$$\lambda_n \quad : \quad \mathcal{E} \longrightarrow R^\infty \quad , \quad E \rightarrow \{\lambda_n[E]\}. \tag{3}$$

For every n, $\lambda_n[E]$ is invariant under the gauge group action on the tetrads [5]. In general, λ_n do not form a set of coordinates neither on the space of gauge orbits nor on the phase space [5].

In order to analyse the phase space further, one has to define the Poisson structure on the set of the eigenvalues. This can be achieved by constructing firstly the symplectic two-form of general relativity [13]

$$\Omega(X,Y) = \frac{1}{4}\int_\Sigma d^3\sigma n_\rho [X^a_\mu, \overleftrightarrow{\nabla}_\tau Y^b_\nu]\epsilon^\tau_{abv}\epsilon^{v\rho\mu\nu}, \tag{4}$$

where $X^a_\mu[E]$ define a vector field on the phase space and the brackets are given by

$$[X^a_\mu, \overleftrightarrow{\nabla}_\tau Y^b_\nu] = X^a_\mu \nabla_\tau Y^b_\nu - Y^a_\mu \nabla_\tau X^b_\nu. \tag{5}$$

Here, Σ is an arbitrary Arnowitt-Deser-Misner surface an n_ρ is its normal one form. The two-form Ω is invertible only on the space of gauge fixed fields since it is degenerate on the space of the solutions of the Einstein equations. The coefficients of Ω are given by the following relation

$$\Omega^{\mu\nu}_{IJ}(x,y) = \int_\Sigma d^3\sigma \; n_\rho \; [\delta(x,x(\sigma))\overleftrightarrow{\nabla}_\tau \delta(y,x(\sigma))] \; \epsilon^\tau{}_{IJv}\,\epsilon^{v\rho\mu\nu}. \tag{6}$$

As was already noted in [6], the symplectic two-form (4) can be written in terms of the Dirac eigenvalues if the map (3) is locally invertible on the phase space. Then, the coefficients Ω_{mn} of Ω defined by the following relation

$$\Omega = \Omega_{mn} d\lambda_n \wedge d\lambda_m, \tag{7}$$

can be expressed in terms of (6) as follows

$$\Omega_{mn}\ T_{n}{}_{I}^{\mu}(x)\ T_{m}{}_{J}^{\nu}(y) = \Omega^{\mu\nu}_{IJ}(x,y), \tag{8}$$

where

$$T_{n}{}_{I}^{\mu}(x) = \frac{\delta\lambda_n[E]}{\delta E^I_\mu(x)}. \tag{9}$$

In order to have a complete description of the phase space of the theory in terms of the Dirac eigenvalues, one has to express the coefficients Ω_{mn} in terms of $\Omega^{\mu\nu}_{IJ}(x,y)$, that is to invert (9). To this end, we introduce the following objects which are well defined since the map (3) is invertible (a necessary condition for the existence of Ω_{mn})

$$U_{n}{}_{\mu}^{I}(x) = \frac{\delta E^I_\mu(x)}{\delta\lambda_n}. \tag{10}$$

A simple algebra shows that the following two relations hold

$$\begin{aligned} U_{n}{}_{\mu}^{I}(x) T_{n}{}_{j}^{\nu}(y) &= \delta^I_J \delta^\nu_\mu \delta^{(4)}(x-y), &(11)\\ U_{n}{}_{\mu}^{I}(x) T_{m}{}_{I}^{\mu}(x) &= \delta_{mn}. &(12)\end{aligned}$$

Note that the coefficients Ω_{mn} do not depend explicitely on the point x of M. Moreover, since the eigenvalues λ_n of D are defined globally on M, Ω_{mn} has the same property. Therefore, in order to elliminate the dependence on the points x and y, one has to integrate twice over M when inverting (8). Then, using (10) and (12) one can obtain from (8) the following relation

$$\Omega_{mn} = \frac{1}{V_M^2} \int_M Dx \int_M Dy\ \Omega^{\mu\nu}_{IJ}(x,y)\ U_{n}{}_{\mu}^{I}(x)\ U_{m}{}_{\nu}^{I}(y), \tag{13}$$

where $Dx = d^4x\sqrt{g}$ and V_M is the four-volume of M. Note that in order to obtain the relation (8) from (13) one has either to rescale the relation (12) by a factor of V_M in the r.h.s. or to rescale the delta-function integral on M. In what follows, we are going to use the relation

$$\int_M Dx\ f(x)\delta^{(4)}(x-y) = V_M f(y). \tag{14}$$

A formal solution of (13) can be given once $U_{n}{}_{\mu}^{I}(x)$'s are known. To calculate them, we use the Dirac equation (1). Assume that the Dirac eigenspinors satisfy the global orthoganality and closure relations

$$\begin{aligned} \langle\psi_n \mid \psi_m\rangle &= \delta_{nm}, &(15)\\ \sum_n \mid \psi_n\rangle\langle\psi_n\mid &= \mathbb{I}, &(16)\end{aligned}$$

where the scalar product in the Hilbert space of the vector fields on M is defined as

$$\langle\psi\mid\phi\rangle=\int_M Dx\ \langle\psi(x)\mid\phi(x)\rangle. \tag{17}$$

Here, $\langle\psi(x)\mid\phi(x)\rangle$ is the scalar product in the local spinor fiber $S_x(M)$ over x. The local spinor sections $\{\mid\psi_n(x)\rangle\}$ are induced by the fields $\{\mid\psi_n\rangle\}$. We assume further that the global fields are defined by integral of local spinors

$$\mid\psi_m\rangle=\int_M Dx\ \mid\psi_m(x)\rangle. \tag{18}$$

It is worth to notice that the relation above implies the following bilocal action of the Dirac operator

$$D(x)\mid\psi_m(y)\rangle=V_M^{-1}\delta^{(4)}(x-y)\mid\psi_m(x)\rangle, \tag{19}$$

where the relation (14) was taken into account. Then, the bilocal scalar product and the local closure relations are given by the following relations

$$\begin{aligned}\langle\psi_n(x)\mid\psi_m(y)\rangle &= V_M^{-1}\delta_{nm}\delta^{(4)}(x-y), &(20)\\ \sum_n\mid\psi_n(x)\rangle\langle\psi_n(x)\mid &= V_M^{-1}\mathbb{I}. &(21)\end{aligned}$$

The local orthogonality and closure relations must be defined in order to deal with the local terms in the relation (13).

The next step is to project Ω_{mn} onto the basis formed by the eigenspinors of D. Since the coefficients of the symplectic form in the basis formed by λ_n are globally defined on M, the projection should be performed onto the basis $\{\mid\psi_n\rangle\}$ rather than onto $\{\mid\psi_n(x)\rangle\}$. By using the relations (15), (16), (18), (20) and (21), one can easily show that the components of Ω_{mn} are given by

$$[\Omega_{mn}]_{st}=V_M^2\int Dx\sum_{r,k}[\Omega^{\mu\nu}_{IJ}(x,x)]_{sr}[U_{n}{}^{I}_{\mu}(x)]_{rk}[U_{m}{}^{j}_{\nu}(x)]_{kt}, \tag{22}$$

where we are using the following shorthand notations

$$[\Omega_{mn}]_{st}=\langle\psi_s\mid[\Omega_{mn}]\mid\psi_t\rangle,\qquad[\Omega^{\mu\nu}_{IJ}(x,y)]_{sr}=\langle\psi_s(x)\mid[\Omega^{\mu\nu}_{IJ}(x,y)]\mid\psi_r(y)\rangle, \tag{23}$$

$$[U_{n}{}^{I}_{\mu}(x)]_{rk}=\langle\psi_r(x)\mid[U_{n}{}^{I}_{\mu}(x)]\mid\psi_k(x)\rangle. \tag{24}$$

Given the manifold M, one could calculate, in principle, the matrix elements of $[\Omega^{\mu\nu}_{IJ}(x,x)]$ after computing the spectrum of the Dirac operator. What is left are the matrix entries from (24). To obtain them we derive the local Dirac equation with respect to the eigenvalue λ_m. The resulting relation has the following form

$$V_M^2\sum_k[U_{m}{}^{I}_{\mu}(x)]_{rk}[D^{\mu}_{I}(x)]_{kn}=\delta_{mn}\delta_{rn}, \tag{25}$$

for all m,r and n. Here, the sum is over k only and

$$[D^{\mu}_{I}(x)]_{kn}=\langle\psi_k(x)\mid[D^{\mu}_{I}(x)]\mid\psi_n(x)\rangle. \tag{26}$$

These terms are determined by the eigenspinors of D and by noting that

$$\begin{aligned}
D^\mu_I(x) &= i\gamma^J\{-E^\nu_I(x)E^\mu_J(x)(\partial_\nu + \omega_{\nu KL}(x)\sigma^{KL}) \\
&- \frac{1}{2}E^\nu_J(x)[E^\rho_I(x)E^\mu_K(x)(\partial_\nu E_{\rho L}(x) - \partial_\rho E_{\nu L}(x)) + \delta_{LI}\partial_\sigma(E^\mu_K(x)\delta^\sigma_\nu - E^\sigma_K(x)\delta^\mu_\nu) \\
&+ \delta^M_I\delta^\mu_\nu E^\tau_K(x)E^\sigma_L(x) - E^\tau_I(x)E^\mu_K(x)E^\sigma_L(x)E^M_\nu(x) \\
&- E^\tau_K(x)E^\sigma_I(x)E^\mu_L(x)E^M_\nu(x))\partial_\sigma E_{\tau M}(x) \\
&- \partial_\rho(E^\mu_K(x)E^\rho_L(x)E_{\nu I}(x)) - (K \leftrightarrow L)]\sigma^{KL}\},
\end{aligned} \tag{27}$$

where $\omega_{\mu IJ}(x)$ are the components of the spin-connection in the spin bundle $S(M)$ over M, γ^I are the tangent-space Dirac matrices and $\sigma^{IJ} = \frac{1}{4}[\gamma^I, \gamma^J]$. In principle, one can compute the matrix elements of (27) if the vielbein is fixed and the eigenspinors of D are known. Therefore, one has to solve the system (25) in order to find $[\Omega_{mn}]_{st}$.

In general, the Dirac operator may have an infinite set of eigenspinors on M, which makes the system (25) infinite, too. A necessary and sufficient condition for the determinant $\det[D^\mu_I(x)]_{kn}$ be absolutely convergent [14] is that the product $\prod_k |[D^\mu_I(x)]_{kk}|$ converges absolutely and there is a non-negative integer number p such that

$$\sum k[\sum_l |[D^\mu_I(x)]_{kl}|^p]^{\frac{1}{p-1}} \ , \quad k \neq l \tag{28}$$

be convergent, too. Let us assume that this is the case. Then the solution of the system (25) is determined by the convergence of the determinant of $[D^\mu_I(x)]_{kn}$. If it converges to zero, the elements of the matrices $V^2_M[U_m{}^I_\mu(x)]$ may exist, but are undetermined. A more interesting case is given by a non-zero determinant of $[D^\mu_I(x)]_{kn}$. Then the inverse of this matrix can be constructed. If we put $m = n$ in (25), we see that some of the elements of the full set of matrices $\{V^2_M[U_m{}^I_\mu(x)]\}$ are proportional to the elements of the inverse of $[D^\mu_I(x)]_{kn}$. Actually, the condition $m = n$ determines the following elements of the matrices $\{V^2_M[U_m{}^I_\mu(x)]\}$

$$[U_n{}^I_\mu(x)]_{nk} = V^{-2}_M[D^\mu_I(x)]^{-1}_{nk}, \tag{29}$$

for all k. Here, $[D^\mu_I(x)]^{-1}_{kn}$ represent the elements of the inverse of $[D^\mu_I(x)]_{kn}$. These elements exist if the complement of each element of $[D^\mu_I(x)]$ converges. The condition that $[U_n{}^I_\mu(x)]_{nk}$ form a matrix (more exactly, the condition that the matrix product between $[U_n{}^I_\mu(x)]_{nk}$ and the inverse $[D^\mu_I(x)]^{-1}_{nk}$ be well defined) leads to the following supplementary relations:

$$[U_s{}^I_\mu(x)]_{sk} = [U_1{}^I_\mu(x)]_{sk} = [U_2{}^I_\mu(x)]_{sk} = \cdots, \tag{30}$$

for all k. The relations (29) and (30) show that the set of the matrices $\{V^2_M[U_m{}^I_\mu(x)]\}$ is degenerate and that these matrices should be proportional to the inverse of the matrix of expected values of the Dirac operator. In the case when $m \neq n$ the r.h.s of (25) vanishes. Therefore, if the determinant of $[D^\mu_I(x)]_{kn}$ does not vanishes, the corresponding matrix elements should be zero. We note that the two cases are incompatible with each other, since the same matrix elements of $V^2_M[U_m{}^I_\mu(x)]$ enter both of them and, while in the case $m = n$ they are determined in terms of the elements $[D^\mu_I(x)]^{-1}_{nk}$, in the case $m \neq n$ they are either zero or are undetermined.

To conclude, one can formally solve the equation (8) as in (13). If the determinant of $[D^{\mu}_{I}(x)]_{kn}$ is different zero, the terms $[U_{m_{\mu}}{}^{I}(x)]_{kl}$ from (13) are determined by the relations (29) and (30). The matrices $\{V^2_M[U_{m_{\mu}}{}^{I}(x)]\}$ should be all proportional to the inverse $[D^{\mu}_{I}(x)]^{-1}$ and the solution is degenerate in this sense. It is not clear if this degeneracy is related in some way to the fact that we are dealing with the set of smooth vielbeins instead of the gauge fixed ones.

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